Chaotic Dynamics of Coupled Transverse-Longitudinal Plasma Oscillations in Magnetized Plasmas

D. Teychenné,* E. Bésuelle,† A. Oloumi,‡ and R. R. E. Salomaa
Department of Engineering Physics and Mathematics, Helsinki University of Technology, P.O. Box 2200, FIN-02015 HUT, Finland
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The propagation of intense electromagnetic waves in cold magnetized plasma is tackled through a relativistic hydrodynamic approach. The analysis of coupled transverse-longitudinal plasma oscillations is performed for traveling plane waves. When these waves propagate perpendicularly to a static magnetic field, the model is describable in terms of a nonlinear dynamical system with 2 degrees of freedom. A constant of motion is obtained and the powerful classical mechanics methods can be used. A new class of solutions, i.e., the chaotic solutions, is discovered by the Poincaré surface of sections. As a result, coupled transverse-longitudinal plasma oscillations become aperiodically modulated.

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An intense electromagnetic field can drive an unmagnetized electron plasma into relativistic regime and generate longitudinal plasma waves due to the self-consistent strong Lorentz force $\mathbf{v} \times \mathbf{B}$. Thus, the transverse-longitudinal oscillations of electron plasma are coupled. In the past, this problem has been intensively studied in the case of a homogeneous, cold, collisionless, unmagnetized plasma (see, e.g., Decoster [1] and references therein). Originally, the analysis in the relativistic regime was performed by Akhiezer and Polovin [2]; later on the results were applied to laser-plasma interaction by Kaw and Dawson [3]. The problem was revisited using a Hamiltonian description by Kaw *et al.* [4]. These studies (without external magnetic field $\mathbf{B}_s$) have revealed that the problem of coupled nonlinear waves can be reduced to a Hamiltonian system having traveling wave solutions which depend only on a single variable $\eta = (\omega t - k x)$, where the angular frequency and the wave vector are ω and k, respectively. A large variety of nonlinear wave solutions exist, but no chaotic solutions appear even in the strongly nonlinear regime.

In the classical regime, the propagation of an electromagnetic wave through a magnetized plasma involves a rich variety of propagation modes due to the coupling term $\mathbf{v} \times \mathbf{B}_s$ [5,6]. Although, this problem has been formulated using a Hamiltonian description [1], it has never been analyzed in such detail as the nonmagnetized relativistic fluid model of Ref. [2]. This present paper describes for the first time the effects of a static magnetic field on the coupled transverse-longitudinal plasma oscillations in the relativistic regime. In particular, the existence of a new class of solutions, i.e., the chaotic solutions, is revealed with the Poincaré surface of sections.

The motion of a single electron can turn chaotic in the presence of both an electromagnetic field and an external static magnetic field [7,8]. Recently, these findings were verified also for collective electron motion in magnetized plasmas [9–11]. It is, however, important to note that in these studies the waves were always considered as driven waves, and their propagations in the plasma were taken into account through the dispersion relation. The Maxwell equations were not solved simultaneously with the fluid equations. Therefore, the possible nonlinear effects on the wave propagation were not properly taken into account.

The system of coupled transverse-longitudinal waves is described self-consistently by solving the Maxwell equations together with the relativistic fluid equations, in the presence of a static magnetic field $\mathbf{B}_s$. We assume a cold, collisionless, homogeneous plasma. The basic equations read

$$\begin{cases} \Delta\phi + \dfrac{\partial}{\partial t}\nabla\cdot\mathbf{A} = -\dfrac{\rho}{\epsilon_0}, \\ \Delta\mathbf{A} - \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\nabla\cdot\mathbf{A} + \dfrac{1}{c^2}\dfrac{\partial\phi}{\partial t}\right) = -\mu_0\mathbf{J}, \end{cases} \tag{1}$$

$$\frac{d\mathbf{p}}{dt} = e\left(\frac{\partial\mathbf{A}}{\partial t} + \nabla\phi - \frac{\mathbf{p}}{\gamma m_e} \times (\nabla\times\mathbf{A} + \mathbf{B}_s)\right), \tag{2}$$

where $\rho = -e(n_e - n_{e0})$ and $\mathbf{J} = -en_e\mathbf{p}/(\gamma m_e)$ are the charge and current densities, respectively. The ions form an immobile neutralizing background ($Zn_i = n_{e0}$); n_e is the electron density, e and m_e are, respectively, the electron charge and mass. The vector and scalar potentials are denoted by $\mathbf{A}$ and ϕ, respectively. The relativistic momentum $\mathbf{p} = \gamma m_e\mathbf{v}$ includes the Lorentz factor $\gamma = 1/\sqrt{1 - \|\mathbf{v}\|^2/c^2}$.

We consider an electromagnetic wave of arbitrary amplitude propagating in the $\hat{\mathbf{x}}$ direction and polarized perpendicularly to the external static magnetic field $\mathbf{B}_s = B_s\hat{\mathbf{z}}$ applied to the plasma. The wave propagates in extraordinary mode: the wave are partially transverse and partially longitudinal. In the Coulomb gauge ($\nabla\cdot\mathbf{A} = 0$), the term $\nabla\times\mathbf{A}$ represents the wave magnetic field. The electric field components of such a wave are given by $E_y = -\partial A_y/\partial t$ and $E_x = -\partial\phi/\partial x$ in transverse and longitudinal directions, respectively. The motion of the

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electron plasma is confined to the xy plane. The motion along $\hat{\mathbf{z}}$ is decoupled and is thus neglected here. We search traveling wave solutions which depend only on the independent variable $\eta = (\omega t - kx)$. The phase velocity $v_\varphi = \omega/k$ is assumed always greater than the speed of light c. Thus, the wave-particle interactions and wave breaking are avoided and the hydrodynamic description is justified even in the strong field limit.

We introduce the following normalized variables:

$$\begin{cases} \tau = \omega t;\ \xi = \dfrac{\omega}{c}x;\ P_\parallel = \dfrac{p_x}{m_e c};\ P_\perp = \dfrac{p_y}{m_e c};\ n_0 = \dfrac{n_{e0}}{n_c}; \\ \beta_\varphi = \dfrac{v_\varphi}{c};\ \eta = \dfrac{\sqrt{n_0}\,\beta_\varphi}{\sqrt{\beta_\varphi^2 - 1}}\left(\tau - \dfrac{\xi}{\beta_\varphi}\right);\ \Omega = \dfrac{\omega_c}{\omega_p}, \end{cases} \tag{3}$$

where $n_c = (\omega^2 m_e \varepsilon_0)/e^2$ is the critical density associated to the angular frequency ω of the wave. The plasma frequency and the cyclotron frequency are denoted by $\omega_p = (n_{e0}\, e^2/m_e \varepsilon_0)^{1/2}$ and $\omega_c = eB_s/m_e c$, respectively. Eliminating the scalar and vector potentials from Eqs. (1) and (2), and performing the further transformation $Y = P_\perp\sqrt{\beta_\varphi^2 - 1}$ and $X = (\beta_\varphi P_\parallel - \sqrt{1 + P_\perp^2 + P_\parallel^2})$, we find

$$\begin{cases} \dfrac{d^2Y}{d\eta^2} = -\dfrac{\beta_\varphi Y}{D} + \Omega\dfrac{d}{d\eta}\left(\dfrac{\beta_\varphi X}{D}\right), \\ \dfrac{d^2X}{d\eta^2} = -\dfrac{\beta_\varphi X}{D} - 1 - \Omega\dfrac{d}{d\eta}\left(\dfrac{\beta_\varphi Y}{D}\right), \end{cases} \tag{4}$$

where $D = \sqrt{\beta_\varphi^2 - 1 + Y^2 + X^2}$. Equations (4) represent a conservative system with 2 degrees of freedom where η plays the role of "time." Equations (4) can also be obtained from Eqs. (12) and (13) from Akhiezer *et al.* [2]. The coupled transverse-longitudinal oscillations of plasma are strictly equivalent to a classical motion of a fictitious particle of unit mass in a two-dimensional potential.

The first integral $I(P_X, P_Y, X, Y)$ or isolating integral of the dynamical system (4) has been discovered. It means the total energy E_0—the sum of the wave energy and the kinetic particle energy—conserves with time and reads

$$I = \frac{1}{2}\left[P_X + \Omega\left(\frac{\partial V}{\partial Y}\right)\right]^2 + \frac{1}{2}\left[P_Y - \Omega\left(\frac{\partial V}{\partial X}\right)\right]^2 + V = E_0, \tag{5}$$

where $V(X, Y, \beta_\varphi) = \beta_\varphi D + X$. The effective space variables are (X, Y) and $(P_X = \dot{X}, P_Y = \dot{Y})$, the longitudinal and transverse momenta, respectively. Using the field variables, $\mathbf{E}$ and $\mathbf{B}$, it is possible to show from Eq. (5) that $dI/dt = 0$ gives the Poynting theorem [12]: the self-consistent solution of the electromagnetic fields and particle dynamics without a dissipative term makes the problem conservative. The invariant I is very similar to a Hamiltonian for classical motion of a particle in an electromagnetic field. But it is written in noncanonical variables and does not satisfy the Hamilton equations ($\dot{p} = -\partial H/\partial q, \dot{q} = \partial H/\partial p$). However, the existence of the first integral, I, allows us to study the solutions of the dynamical system (4) by using the Poincaré surface of sections.

The invariant I involves three parameters: the total energy E_0, the phase velocity β_φ, and $\Omega = \omega_c/\omega_p$, the ratio of the cyclotron to the plasma frequencies. At the equilibrium point $(X, Y) = (-1, 0)$, where the forces and the momenta cancel ($\dot{P}_X = \dot{P}_Y = P_X = P_Y = 0$), the minimum energy is given by $E_{\min} = (\beta_\varphi^2 - 1)$ which corresponds to the nonoscillating solution. Scanning on the parameters of I, it is possible to analyze in detail the different solutions of Eqs. (4). The total energy E_0, i.e., the constant of motion I, and the phase velocity β_φ are fixed, and thereafter we construct for different values of the parameter Ω the Poincaré surface of sections by plotting P_X vs X each time $Y = 0$ and $P_Y \geq 0$.

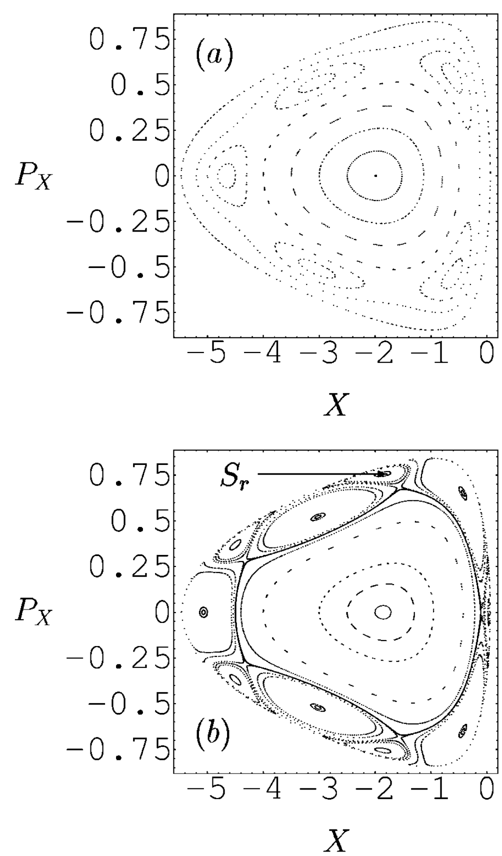

FIG. 1. Poincaré surface of section in (P_X, X) phase space $(P_Y \geq 0, Y = 0)$ for $E_0 = 0.57$ and $\beta_\varphi = 1.1$ when (a) $\Omega = 0$ or (b) $\Omega = 0.15$. No chaotic solutions.

In the absence of the external static magnetic field $\mathbf{B}_s$, ($\Omega = 0$), Eq. (5) reduces to the Hamiltonian given in [2]. Figure 1(a) displays a typical Poincaré surface of section without the external static magnetic field for $E_0 = 0.57$ and $\beta_\varphi = 1.1$. The central fixed point ($P_X = 0, X \approx -2$) corresponds to periodic wave solutions. The higher order fixed points (the five elliptic points) and the islands around them arise from the resonances induced by the coupling. These solutions represent nonlinearly modulated electromagnetic and plasma waves: the wave envelope is modulated periodically (elliptic points) or quasiperiodically (islands).

We discuss next the different solutions of Eqs. (4) for finite magnetic field strengths ($\Omega > 0$). Figure 1(b) shows the Poincaré surface of section for $E_0 = 0.57$, $\beta_\varphi = 1.1$, and $\Omega = 0.15$ in the weakly relativistic regime. Similar solutions to those presented in Fig. 1(a) are found and, in addition, new resonances appear. The magnetic field in Eqs. (4) enhances the coupling and creates a new perturbation in the "particle orbits." Thus, new elliptic points and an "island chain" are produced. These solutions correspond to nonlinearly modulated extraordinary waves. The nonlinearity introduced by the magnetic field generates new harmonics.

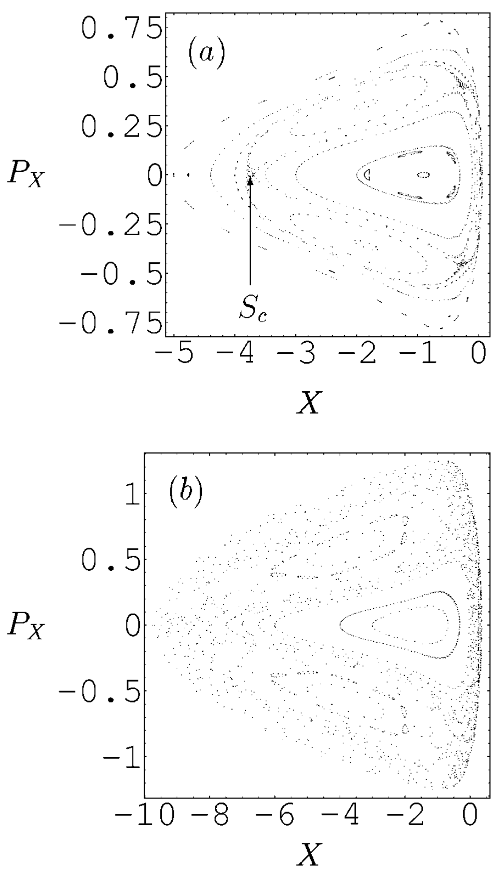

FIG. 2. Poincaré surface of section in (P_X, X) phase space ($P_Y \geq 0, Y = 0$) for $\beta_\varphi = 1.1$ when (a) $E_0 = 0.57$ and $\Omega = 1.5$ or (b) $E_0 = 1.0$ and $\Omega = 1.5$. Chaotic solutions.

In Fig. 2(a) we present the Poincaré surface of section for $E_0 = 0.57$, $\beta_\varphi = 1.1$, and $\Omega = 1.5$. A thin layer of stochasticity appears near the separatrix, i.e., around the hyperbolic points. A new class of solutions is revealed. As a result, the coupled transverse-longitudinal motion of the electron plasma is chaotic. The amplitude of transverse and longitudinal electric fields associated to the extraordinary waves becomes unpredictable and is very sensitive to the initial conditions. From the physical point of view, this chaotic behavior in coupled transverse-longitudinal plasma oscillations implies strong harmonics generation. For $\beta_\phi = 10$ which is a rather high value and $E_0 = 0.57$, $\Omega = 1.5$, chaos disappears and the Poincaré surface of section is similar to the one of a two-dimensional harmonic oscillator. Figure 2(b) shows the Poincaré surface of section for the relativistic regime $E_0 = 1$. Under a strong perturbation, high order resonances are generated and the chaos spreads to a part of the Poincaré surface of section. For the high values of E_0 and Ω, global stochasticity is observed, and only some regular trajectories around the central fixed point are still preserved.

The numerical integration of Eqs. (4) allows us to compute the evolution of the electric fields vs the time η from

$$\begin{cases} \tilde{E}_y = -\dfrac{P_Y}{\sqrt{\beta_\varphi^2 - 1}} + \Omega \dfrac{\beta_\varphi}{\sqrt{\beta_\varphi^2 - 1}}\left(1 + \dfrac{\beta_\varphi X}{D}\right), \\ \tilde{E}_x = -\dfrac{P_X}{\beta_\varphi} - \Omega \dfrac{Y}{D}, \end{cases} \tag{6}$$

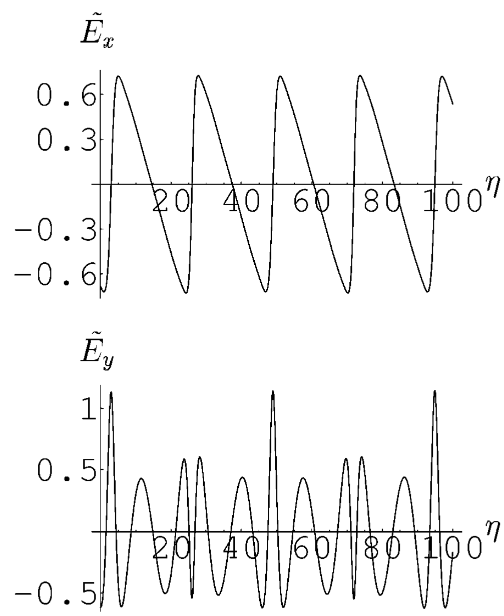

FIG. 3. Longitudinal and transverse electric fields $\tilde{E}_x$ and $\tilde{E}_y$, respectively, as a function of time η. The computation is for the initial condition represented by the point S_r in the Poincaré surface of section, Fig. 1(b), when $\beta_\varphi = 1.1$, $E_0 = 0.57$, and $\Omega = 0.15$. Regular solutions.

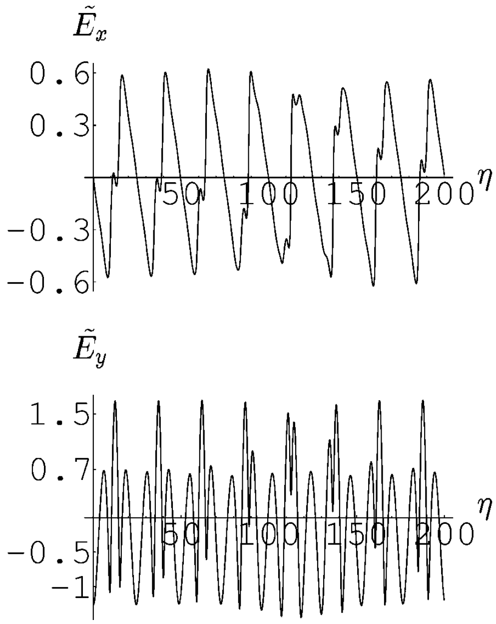

FIG. 4. Longitudinal and transverse electric fields $\tilde{E}_x$ and $\tilde{E}_y$, respectively, as a function of time η. The computation is for the initial condition represented by the point S_c in the Poincaré surface of section, Fig. 2(a), when $\beta_\varphi = 1.1$, $E_0 = 0.57$, and $\Omega = 1.5$. Chaotic solutions.

where the normalized transverse and longitudinal field components are

$$\tilde{E}_y = \frac{\sqrt{\beta_\varphi^2 - 1}}{\beta_\varphi}\left(\frac{eE_y}{m_e \omega_p c}\right)$$

and

$$\tilde{E}_x = \frac{\sqrt{\beta_\varphi^2 - 1}}{\beta_\varphi}\left(\frac{eE_x}{m_e \omega_p c}\right),$$

respectively.

Figure 3 displays the longitudinal and transverse electric fields for a fifth order elliptic point which represents a regular solution when $E_0 = 0.57$, $\beta_\varphi = 1.1$, and $\Omega = 0.15$. The initial conditions represented by the point S_r in the Poincaré surface of section for the computation are shown in Fig. 1(b). The structure of the longitudinal electric field $\tilde{E}_x$ is the same as the one associated to the nonlinear plasma wave. The amplitude of the transverse electric field $\tilde{E}_y$ is anharmonic and modulated.

In Fig. 4, we plot the longitudinal and transverse electric fields corresponding to the chaotic solutions. For $E_0 = 0.57$, $\beta_\varphi = 1.1$, and $\Omega = 1.5$, the initial condition S_c chosen in the Poincaré surface of section is shown in Fig. 2(a). The transverse and longitudinal fields are aperiodically modulated. Their amplitudes become unpredictable and are very sensitive to the initial conditions.

In conclusion, we have studied in a magnetized plasma the propagation of an intense electromagnetic wave in an extraordinary mode. In the case of traveling plane wave solutions, the problem of the coupled transverse-longitudinal plasma oscillations can be reduced to a conservative system. A constant of motion has been found which allows one to use the Poincaré surface of section for the analysis of the different solutions. Depending on the value of the parameters $(E_0, \beta_\varphi, \Omega)$, a large variety of strongly nonlinear waves appear. In particular, a new class of solutions, i.e., the chaotic solutions, has been discovered. In this case, the amplitude of longitudinal and transverse electric fields associated to the waves becomes unpredictable.

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*Email address: denis.teychenne@caramail.com
†Present address: Laboratoire de Physique des Milieux Ionisés, Ecole Polytechnique, F-91128 Palaiseau, France. Email address: emmanuel.besuelle@mail.dotcom.fr
‡Deceased.
Formerly with: National Biocomputation Center, Stanford University Medical Center, Palo Alto, CA.

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